Strichartz type estimates for the damped wave equation and their application (Regularity and Singularity for Partial Differential Equations with Conservation Laws)

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By

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Abstract

In this study, the Cauchy problem for the damped wave equation is considered. Solutions to the homogeneous damped wave equation are decomposed into the heat part and wave part. The decomposition implies that the solution behaves not only as a solution of a heat equation for $t \to \infty$ but also as a solution of a wave equation near $t = 0$. In this paper, based on the above results, we prove time global Strichartz type estimates for two and three dimensions. Furthermore using these estimates, we propose some well-posedness theorems for the nonlinear damped wave equation in the energy class.

§ 1. Introduction

We consider the following inhomogeneous damped wave equation:

$$\text{(DW)} \quad \begin{cases} (\partial_t^2 - \triangle + \partial_t)u(t,x) = h(t,x) \ (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), \end{cases}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\quad \partial_{x_j} = \frac{\partial}{\partial x_j}$ and $\triangle = \sum_{j=1}^d \partial_{x_j}^2$. Our aim is to prove the Strichartz type estimates for (DW) and to apply these estimates in solving the nonlinear problem

$$\text{(NDW)} \quad \begin{cases} (\partial_t^2 - \triangle + \partial_t)u(t,x) = F(u) & (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in W^{1,2} \times L^2, \end{cases}$$

where $F(u) = |u|^{\alpha-1}u$.

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The solution to (DW) is given by

$$u(t) = K(u_0, u_1)(t) + \int_0^t K_1(t-s)h(s)ds,$$

where $K(u_0, u_1)(t) = K_0(t)u_0 + K_1(t)\left(\frac{1}{2}u_0 + u_1\right)$ is the solution to homogeneous problem (DW). Note that the operators $K_j(t)$ $(j = 0, 1)$ are given by the Fourier multipliers

$$\begin{aligned} K_0(t)f &= e^{-\frac{t}{2}}\cos(t\varphi(D))f = e^{-\frac{t}{2}}\mathcal{F}^{-1}\left[\cos(t\varphi(\xi))\mathcal{F}[f]\right], \\ K_1(t)f &= e^{-\frac{t}{2}}\frac{\sin(t\varphi(D))}{\varphi(D)}f = e^{-\frac{t}{2}}\mathcal{F}^{-1}\left[\frac{\sin(t\varphi(\xi))}{\varphi(\xi)}\mathcal{F}[f]\right] \end{aligned}$$

with

$$\varphi(\xi) = \begin{cases} \sqrt{|\xi|^2 - \frac{1}{4}} & (|\xi| > \frac{1}{2}) \\ i\sqrt{\frac{1}{4} - |\xi|^2} & (|\xi| \le \frac{1}{2}) \end{cases}.$$

We know the homogeneous solution $K(u_0, u_1)$ has the following decomposition:

$$K(u_0, u_1)(t) = J(t) + e^{-\frac{t}{2}}W(t),$$

where $W(t)$ is comprised of some solutions to the wave equation and $J(t)$ has the same decay rate as the solution to the heat equation (see [5], [6], [11], [12]). Nishihara [12] shows L^p-L^q estimates of (DW) and applies them to prove global well-posedness for (NDW). Then he obtains the global existence and some decay estimates for $\alpha > 1 + 2/d$ with $(u_0, u_1) \in W^{1,1} \cap W^{1,\infty} \times L^1 \cap L^\infty$. Hosono-Ogawa [5] $(d = 2)$ and Narazaki [11] $(d \ge 2)$ obtain the same L^p-L^q type estimates by Fourier transform and apply it to solve (NDW). Matsumura [9] shows the global well-posedness for several semilinear cases with the small initial data $(u_0, u_1) \in L^1 \cap W^{\left[\frac{d}{2}\right]+2,2} \times L^1 \cap W^{\left[\frac{d}{2}\right]+1,2}$. These results were assumed some additional conditions such as L^p-integrability or smoothness for the initial data. One of the our purpose is to eliminate these additional conditions. There are some results whose initial data belong to the energy class $W^{1,2} \times L^2$. Nakao-Ono [10] prove a local well-posedness for $1 < \alpha < \frac{d+2}{d-2}$ (a global well-posedness for $\frac{d+3}{d-1} < \alpha < \frac{d+2}{d-2}$) with suitably small and compactly supported initial data. Radu [13] obtains the existence of a local weak solution for the general nonlinear damped wave equation with small data which include (NDW) for $1 < \alpha < \frac{d+2}{d-2}$. In this paper, we prove the local well-posedness for $d = 3, 3 < \alpha < 5$ and the global well-posedness for $d = 3, \alpha = 5$ without support compactness of the initial data. Furthermore we prove that the existence time T depends only to $W^{1,2} \times L^2$ norm of (u_0, u_1).

We first prove Strichartz type estimates for the damped wave equation. Strichartz estimates are well known estimates for various dissipative equations. The estimates represent a smoothing effect of the equation and are used for the analysis of linear and nonlinear problems. To obtain a local well-posedness to (NDW), it suffices to show time local Strichartz type estimates. This can be easily shown. Let $v(t,x) = e^{\frac{t}{2}}u(t,x)$, then (DW) becomes the following wave equation with linear term

$$(\partial_t^2 - \triangle)v(t,x) = \frac{1}{4}v(t,x) + e^{\frac{t}{2}}h(t,x).$$

Hence Strichartz estimates for the wave equation (as in [4]) imply time local Strichartz type estimates for u. However, this approach can not give time global Strichartz type estimates. We define that

$$\nabla = (\partial_{x_1}, \cdots, \partial_{x_d}), \quad D = (-i\partial_{x_1}, \cdots, -i\partial_{x_d}),$$

$$\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}, \quad g(D)f = \mathcal{F}^{-1}\left[g(\xi)\mathcal{F}[f]\right],$$

$$W_x^{\mu,p} = \{f; \langle D\rangle^\mu f \in L_x^p\}, \quad \dot{W}_x^{\mu,p} = \{f; |D|^\mu f \in L_x^p\},$$

$$\|f\|_{L_t^\theta \dot{W}_x^{\mu,p}} = \|f\|_{L_t^\theta([0,\infty))\dot{W}_x^{\mu,p}(\mathbb{R}^d)} = \left\{\int_0^\infty \left(\int_{\mathbb{R}^d} ||D|^\mu f|^p dx\right)^{\frac{\theta}{p}} dt\right\}^{\frac{1}{\theta}}$$

and

$$\|f\|_{L_t^\theta W_x^{\mu,p}} = \|f\|_{L_t^\theta([0,\infty))W_x^{\mu,p}(\mathbb{R}^d)} = \left\{\int_0^\infty \left(\int_{\mathbb{R}^d} |\langle D\rangle^\mu f|^p dx\right)^{\frac{\theta}{p}} dt\right\}^{\frac{1}{\theta}},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and Fourier inverse transform defined by

$$\mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$$

and

$$\mathcal{F}^{-1}[g](x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)d\xi.$$

Throughout this paper, we denote $\|\cdot\|_p = \|\cdot\|_{L_x^p(\mathbb{R}^d)}$ and $\|\cdot\|_{W^{\mu,p}} = \|\cdot\|_{W_x^{\mu,p}(\mathbb{R}^d)}$. Hölder conjugate exponent $p \geq 1$ is denoted by p', which satisfies $1/p + 1/p' = 1$. In this paper, we prove the time global Strichartz type estimates for (DW).

Theorem 1.1. *Let $d = 2$ or 3. Assuming (θ_j, p_j) $(j = 1,2)$ satisfy*

$$2 < \theta_j \leq \infty, \quad \begin{cases} 2 \leq p_j \leq \frac{2d}{d-2} & (d=3) \\ 2 \leq p_j < \infty & (d=2) \end{cases}, \tag{1.1}$$

$$d\left(\frac{1}{2} - \frac{1}{p_j}\right) \geq \frac{2}{\theta_j}, \tag{1.2}$$

$$2d\left(\frac{1}{2} - \frac{1}{p_j}\right) - \mu_j \leq \frac{2}{\theta_j}, \tag{1.3}$$

where

$$\mu_j = \max\left\{1, (d+1)\left(\frac{1}{2} - \frac{1}{p_j}\right)\right\} \tag{1.4}$$

(see Figure 1 and Figure 2*). Then the following* (I) *and* (II) *hold,*

(I) *We assume that there exists* $2 \le l \le \frac{2d}{d-1}$ *and* $\sigma \in [0,1]$ *such that*

$$\frac{1}{p_1} \le \frac{1}{l}, \quad \frac{1}{p_1} = \frac{\sigma}{p_2} + \frac{1-\sigma}{l} \quad \textit{and} \quad \frac{1}{\theta_1} = \frac{\sigma}{\theta_2}. \tag{1.5}$$

Then it holds

$$\left\| \int_0^t K_1(t-s)h(s)ds \right\|_{L_t^{\theta_1} L_x^{p_1}} \le C\|h\|_{L_t^{\theta_2'} W_x^{\mu_2-1,p_2'}}. \tag{1.6}$$

(II) *We assume that there exists* $\sigma \in [0,1]$ *such that*

$$\frac{1}{p_2} = \frac{\sigma}{p_1} + \frac{1-\sigma}{2} \quad \textit{and} \quad \frac{1}{\theta_2} = \frac{\sigma}{\theta_1}. \tag{1.7}$$

Then it holds

$$\left\| \int_0^t K_1(t-s)h(s)ds \right\|_{L_t^{\theta_1} L_x^{p_1}} \le C\|h\|_{L_t^{\theta_2'} W_x^{\sigma(\mu_1-1),p_2'}}. \tag{1.8}$$

In (1.6) *and* (1.8)*, the constants* C *depends on* p_j, θ_j, σ *and* d*.*

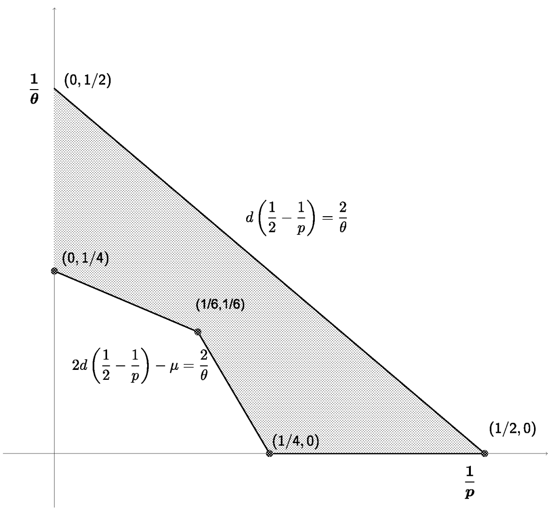

Figure 1. $d = 2$

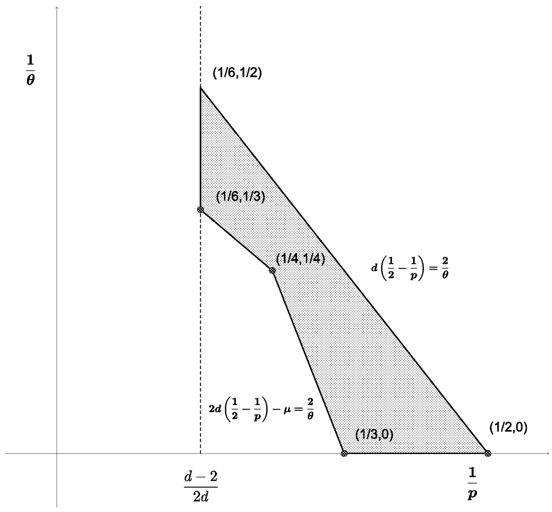

Figure 2. $d = 3$

We note that the condition (1.5) implies that $\frac{1}{l} \geq \frac{1}{p_1} \geq \frac{1}{p_2}$, and the three points $\left(\frac{1}{l}, 0\right)$ and $\left(\frac{1}{p_j}, \frac{1}{\theta_j}\right)$ are on a line. Similarly, condition (1.7) means that $\frac{1}{2} \geq \frac{1}{p_2} \geq \frac{1}{p_1}$, and the three points $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{1}{p_j}, \frac{1}{\theta_j}\right)$ belong a line. Condition (1.2) relates to the Schrödinger equation. Indeed, a pair (θ, p) on the line $d\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{2}{\theta}$ is called L^2-Strichartz pair for Schrödinger equations (as in [3]). Moreover, conditions (1.3) and (1.4) relate to the wave equation. In particular, if we take $\left(\frac{1}{p_j}, \frac{1}{\theta_j}\right)$ from the segment connecting $\left(\frac{d-1}{2d}, 0\right)$ and $\left(\frac{d-1}{2(d+1)}, \frac{d-1}{2(d+1)}\right)$, the statement of Theorem 1.1 accords with the Strichartz estimates for the wave equation (as in [4]).

In this paper, we call u as a solution to (NDW) if u satisfies

$$u(t) = K(u_0, u_1)(t) + \int_0^t K_1(t-s)F(u(s))ds.$$

Using Theorem 1.1, we obtain a time local $W^{1,2} \times L^2$ solution to subcritical (NDW).

Theorem 1.2. *Let $d = 3$, $3 < \alpha < 5$ and $\|(u_0, u_1)\|_{W^{1,2} \times L^2} \leq A$. Then there exists a $T = T(A, \alpha) > 0$ such that* (NDW) *has a unique local solution in*

$$C^1([0,T); L^2) \cap C^0([0,T); W^{1,2}).$$

Note that the theorem is already proved for $\begin{cases} 1 < \alpha < \infty, & (d=2) \\ 1 < \alpha \leq d/(d-2), & (d \geq 3) \end{cases}$ (see [14]). We also prove global well-posedness in the critical case.

Theorem 1.3. *Assume $d = 3$, $\alpha = 5$ and $0 < T \leq \infty$. Then there exists a $\delta > 0$ such that if $(u_0, u_1) \in W^{1,2} \times L^2$ satisfies*

$$\|K(u_0, u_1)\|_{L_t^8([0,T); L_x^8)} \leq \delta,$$

then there exists a unique solution to (NDW) *in*

$$C^1([0,T); L^2) \cap C^0([0,T); W^{1,2}).$$

Using Theorem 1.3, we have the following local and global existence results:

Theorem 1.4. *Assume $d = 3$, $\alpha = 5$ and $(u_0, u_1) \in W^{1,2} \times L^2$. Then the following* (i) *and* (ii) *hold.*

(i) *There exists $T > 0$ such that* (NDW) *has a unique local solution in*

$$C^1([0,T); L^2) \cap C^0([0,T); W^{1,2}).$$

(ii) *There exists $\tilde{\delta} > 0$ such that if $\|(u_0, u_1)\|_{W^{1,2}\times L^2} \le \tilde{\delta}$ then* (NDW) *has a unique global solution in*

$$C^1([0,\infty); L^2) \cap C^0([0,\infty); W^{1,2}).$$

Chen-Fan-Zhang [7] prove the Strichartz estimates for damped fractional wave equations $(\partial_t^2 + (-\triangle)^\gamma + 2\partial_t)u = 0$ and apply them for nonlinear ploblems. They obtain the sharp estimates for the homogeneous part and prove the global well-posedness for nonlinear ploblem with $(u_0, u_1) \in \cap_{1<p<\infty} W^{1,p} \times L^p$. On the other hand Theorem 1.1 is the Strichartz estimates for the nonhomogeneous part and Theorem 1.2-1.4 are only assumed $(u_0, u_1) \in W^{1,2} \times L^2$.

§ 2. Preliminaries

First, we consider the linear wave equation

$$\begin{cases} (\partial_t^2 - \triangle)w(t,x) = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (w(0), \partial_t w(0)) = (w_0, w_1). \end{cases}$$

The solution is given by

$$w(t) = W_0(t)w_0 + W_1(t)w_1,$$

where the operators $W_j(t)$ $(j = 0, 1)$ are defined by

$$W_0(t) = \cos(t|D|) \quad \textit{and} \quad W_1(t) = \frac{\sin(t|D|)}{|D|}.$$

The following properties of W_0 and W_1 are well known.

Proposition 2.1 (L^p-$L^{p'}$ estimates for wave equation(see [2]))**.** *Assume $2 \le p < \infty$ and $(d+1)\left(\frac{1}{2} - \frac{1}{p}\right) \le \mu < d$. Then it holds that*

$$\|W_0(t)g\|_p \le Ct^{-\gamma_W} \|g\|_{\dot{W}^{\mu,p'}} \quad (|t| \ne 0)$$

and

$$\|W_1(t)g\|_p \le Ct^{-\gamma_W} \|g\|_{\dot{W}^{\mu-1,p'}} \quad (|t| \ne 0),$$

where

$$\gamma_W = 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \mu \ge 0.$$

Proposition 2.2 (Strichartz estimates for wave equation(see [4] Proposition 3.1))**.** *Assume $2 \le \theta, p < \infty$ and $\mu \in \mathbb{R}$ satisfy*

$$\left(\frac{2}{\theta}, (d-1)\left(\frac{1}{2} - \frac{1}{p}\right)\right) \ne (1,1),$$

$$\frac{2}{\theta} \le \min\left\{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right), 1\right\}$$

and

$$d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{\theta}=\mu.$$

Then it holds that

$$\|W_0 g\|_{L_t^\theta L_x^p} \le C\|g\|_{W^{\mu,2}}$$

and

$$\|W_1 g\|_{L_t^\theta L_x^p} \le C\|g\|_{W^{\mu-1,2}}.$$

Secondly, we state some properties of (DW) (see [5], [11], [12]). We define Φ_j $(j = 0,1)$ and the norm $\|\cdot\|_E$ as

$$\Phi_j[h](t) = \int_0^t K_j(t-s)h(s)ds \quad (j=0,1)$$

and

$$\|u\|_E^2 = \|u\|^2_{L_t^\infty W_x^{1,2}} + \|\partial_t u\|^2_{L_t^\infty L_x^2} + \|\nabla u\|^2_{L_t^2 L_x^2} + \|\partial_t u\|^2_{L_t^2 L_x^2}. \tag{2.1}$$

The following estimate is obtained by standard energy method.

Proposition 2.3 (Energy dissipative estimate). *Let* $(u_0,u_1) \in W^{1,2}\times L^2$ *and* u *be a solution to* (DW). *Then there exists a constant* $C>0$ *such that for any* $1 \le p, \theta \le \infty$ *and* $\nu \in \mathbb{R}$ *satisfy*

$$\|u\|_E^2 \le C\|(u_0,u_1)\|^2_{W^{1,2}\times L^2} + C\left(\|u\|_{L_t^\theta W_x^{-\nu,p}} + \|\partial_t u\|_{L_t^\theta W_x^{-\nu,p}}\right)\|h\|_{L_t^{\theta'} W_x^{\nu,p'}}. \tag{2.2}$$

Proof. Taking L_x^2 inner product (DW) by $\partial_t u$ and integrating it over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2}\left\{\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2\right\} + \int_0^t \|\partial_t u(s)\|_2^2 ds \\
&\le \|(u_0,u_1)\|^2_{W^{1,2}\times L^2} + \int_0^t \int_{\mathbb{R}^d} \partial_t u(t,x)h(t,x)dxds \\
&\le \|(u_0,u_1)\|^2_{W^{1,2}\times L^2} + \|\partial_t u\|_{L_t^\theta W_x^{-\nu,p}}\|h\|_{L_t^{\theta'} W_x^{\nu,p'}}.
\end{aligned}$$

Similarly, taking L_x^2 inner product (DW) by u and integrating it over $[0,t]$, we have

$$\begin{aligned}
&-\|\partial_t u(t)\|_2^2 + \frac{1}{4}\|u(t)\|_2^2 - \int_0^t \|\partial_t u(s)\|_2^2 ds + \int_0^t \|\nabla u(s)\|_2^2 ds \\
&\le \|(u_0,u_1)\|^2_{W^{1,2}\times L^2} + \|u\|_{L_t^\theta W_x^{-\nu,p}}\|h\|_{L_t^{\theta'} W_x^{\nu,p'}}.
\end{aligned}$$

Combining the above estimates, we obtain

$$\|u(t)\|_{W^{1,2}}^2+\|\partial_t u(t)\|_2^2+\int_0^t\|\nabla u(s)\|_2^2ds+\int_0^t\|\partial_t u(s)\|_2^2ds$$
$$\le C\|(u_0,u_1)\|_{W^{1,2}\times L^2}^2+C\left(\|u\|_{L_t^\theta W_x^{-\nu,p}}+\|\partial_t u\|_{L_t^\theta W_x^{-\nu,p}}\right)\|h\|_{L_t^{\theta'}W_x^{\nu,p'}}\quad(t\ge 0),$$

which is (2.2). This completes the proof of Proposition 2.3. □

It is known that u has the following L^p-$L^{q'}$ estimates.

Proposition 2.4 (L^p-$L^{q'}$ estimates for the damped wave equation (see [5],[12])). *Let $d=2$ or 3. Assume $1\le q'\le p\le\infty$. Then it holds that*

$$\left\|\left\{K_0(t)-e^{-\frac{t}{2}}\left(W_0(t)+\frac{t}{8}W_1(t)\right)\right\}g\right\|_p\le C(1+t)^{-\gamma_D-1}\|g\|_{q'}\quad(t\ge 0)\tag{2.3}$$

and

$$\left\|\left\{K_1(t)-e^{-\frac{t}{2}}W_1(t)\right\}g\right\|_p\le C(1+t)^{-\gamma_D}\|g\|_{q'}\quad(t\ge 0),\tag{2.4}$$

where

$$\gamma_D=\frac{d}{2}\left(\frac{1}{q'}-\frac{1}{p}\right).\tag{2.5}$$

Proposition 2.4 means that the remainder given by subtracting the wave part from u has L^p-$L^{q'}$ decay estimates like solutions to the heat equations. Furthermore the remainder is bounded near $t=0$. Thus we can understand that the singularity of the solutions to (DW) in $t=0$ come from the wave part.

Finally, we introduce some other known results. The following three Propositions are well known (see for example [15] Chapter 0).

Proposition 2.5 (Sobolev embedding). *Let $1<r_1,r_2<\infty$ and $s_1<s_2$. Then there exists a constant $C>0$ such that*

$$\|f\|_{\dot W^{s_1,r_1}}\le C\|f\|_{\dot W^{s_2,r_2}}$$

whenever $s_1-\frac{d}{r_1}=s_2-\frac{d}{r_2}$, and

$$\|f\|_{W^{s_1,r_1}}\le C\|f\|_{W^{s_2,r_2}}$$

whenever $s_1-\frac{d}{r_1}\le s_2-\frac{d}{r_2}$.

Proposition 2.6 (Hardy-Littlewood-Sobolev inequality). *Suppose $2<\theta<\infty$. Then there exists a constant $C>0$ such that*

$$\left\|\frac{1}{|\cdot|^{\frac{2}{\theta}}}*f\right\|_{L_t^\theta}\le C\|f\|_{L_t^{\theta'}}.$$

Proposition 2.7 (Young's inequality). *Let $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} \ge 1$. Then it holds that*

$$\|f * g\|_{L^r} \le \|f\|_p \|g\|_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

The next interpolation lemma plays an important role in this paper.

Lemma 2.8. *Let θ, p satisfy*

$$2 \le \theta \le \infty, \quad \begin{cases} 2 \le p \le \frac{2d}{d-2} & (d \ge 3) \\ 2 \le p < \infty & (d = 2) \end{cases} \quad \text{and} \quad d\left(\frac{1}{2} - \frac{1}{p}\right) \ge \frac{2}{\theta}. \tag{2.6}$$

Then there exists a constant $C > 0$ such that

$$\|f\|_{L_t^\theta L_x^p} \le C\|f\|_E. \tag{2.7}$$

In particular it holds that

$$\|K(u_0, u_1)\|_{L_t^\theta L_x^p} \le C\|(u_0, u_1)\|_{W^{1,2} \times L^2}. \tag{2.8}$$

Proof. We can easily prove (2.8) by (2.7) and Proposition 2.3. Thus we prove (2.7). Let $d \ge 3$. Using Proposition 2.5, we have $\|\cdot\|_{L_x^{\frac{2d}{d-2}}} \le C\|\cdot\|_{\dot{W}_x^{1,2}}$. Thus for any $2 \le \theta \le \infty$, we have

$$\|f\|_{L_t^\theta L_x^{\frac{2d}{d-2}}} \le C\|f\|_{L_t^\theta \dot{W}_x^{1,2}} \le C\{\|f\|_{L_t^\infty \dot{W}_x^{1,2}} + \|f\|_{L_t^2 \dot{W}_x^{1,2}}\} \le C\|f\|_E.$$

On the other hand it is clear that

$$\|f\|_{L_t^\infty L_x^2} \le C\|f\|_E.$$

Interpolating the above estimates by Hölder inequality, we get (2.7).

Next, we suppose $d = 2$. For any θ, p which satisfy $2\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{2}{\theta}$ and $2 \le p < \infty$, by Gagliardo-Nirenberg inequality (e.g. [3] Thm 1.3.7), it holds that

$$\|f\|_{L_x^p} \le C\|f\|_{L_x^2}^{1-\sigma} \|f\|_{\dot{W}_x^{1,2}}^\sigma,$$

where $\sigma \in [0, 1)$ is given by

$$\frac{1}{p} = \frac{1-\sigma}{2}.$$

Note that $\sigma\theta = 2$, it holds that

$$\begin{aligned} \|f\|_{L_t^\theta L_x^p} &\le C\left\| \|f\|_{L_x^2}^{1-\sigma} \|f\|_{\dot{W}_x^{1,2}}^\sigma \right\|_{L_t^\theta} \le C\|f\|_{L_t^\infty L_x^2}^{1-\sigma} \|f\|_{L_t^{\sigma\theta} \dot{W}_x^{1,2}}^\sigma \\ &\le C\{\|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^{\sigma\theta} \dot{W}_x^{1,2}}\} \le C\|f\|_E. \end{aligned}$$

On the other hands, using Proposition 2.5, we have

$$\|f\|_{L_t^\infty L_x^p} \le \|f\|_{L_t^\infty W_x^{1,2}} \le C\|f\|_E.$$

for any $2 \le p < \infty$. Interpolating the above estimates, we get (2.7). This completes the proof of Lemma 2.8. □

§ 3. The Strichartz type estimate

In this section, we prove Theorem 1.1. First, we prove $L_t^\theta L_x^p$-$L_t^1 L_x^2$ type estimate.

Lemma 3.1. *Let θ and p satisfy* (2.6) *that is*

$$2 \le \theta \le \infty, \qquad \begin{cases} 2 \le p \le \frac{2d}{d-2} & (d=3) \\ 2 \le p < \infty & (d=2) \end{cases} \quad \text{and} \quad d\left(\frac{1}{2} - \frac{1}{p}\right) \ge \frac{2}{\theta}.$$

Then it holds that

$$\|\Phi_1[h]\|_{L_t^\theta L_x^p} \le C\|h\|_{L_t^1 L_x^2}. \tag{3.1}$$

Proof. We remark that $v = \Phi_1[h]$ is a solution to (DW) with $(u_0, u_1) = (0, 0)$. Hence using Proposition 2.3 for $(\theta, p) = (\infty, 2)$ and $\nu = 0$, we obtain

$$\begin{aligned} \|v\|_E^2 &\le C\{\|v\|_{L_t^\infty L_x^2} + \|\partial_t v\|_{L_t^\infty L_x^2}\}\|h\|_{L_t^1 L_x^2} \\ &\le \frac{1}{2}\|v\|_{L_t^\infty L_x^2}^2 + \frac{1}{2}\|\partial_t v\|_{L_t^\infty L_x^2}^2 + C\|h\|_{L_t^1 L_x^2}^2 \le \frac{1}{2}\|v\|_E^2 + C\|h\|_{L_t^1 L_x^2}^2. \end{aligned}$$

This estimate means

$$\|v\|_E \le C\|h\|_{L_t^1 L_x^2}.$$

From Lemma 2.8, it holds (3.1). This completes the proof of Lemma 3.1. □

Next, we prove $L_t^\theta L_x^p$-$L_t^{\theta'} L_x^{p'}$ type estimates.

Lemma 3.2. *Suppose* $2 \le p < \infty$, $2 < \theta < \infty$ *satisfy*

$$\gamma_D = \frac{d}{2}\left(1 - \frac{2}{p}\right) \ge \frac{2}{\theta}, \tag{3.2}$$

$$\gamma_W = 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \mu \le \frac{2}{\theta}, \tag{3.3}$$

where

$$\mu = \max\left\{1, (d+1)\left(\frac{1}{2} - \frac{1}{p}\right)\right\}. \tag{3.4}$$

Then it holds that

$$\|\Phi_0[h]\|_{L_t^\theta L_x^p} \le \|h\|_{L_t^{\theta'} W_x^{\mu,p'}} \tag{3.5}$$

and

$$\|\Phi_1[h]\|_{L_t^\theta L_x^p} \le \|h\|_{L_t^{\theta'} W_x^{\mu-1,p'}}. \tag{3.6}$$

Proof. First, we prove (3.6). Using Proposition 2.1 and Proposition 2.4 for $q = p$, we have

$$\begin{aligned}
\|\Phi_1[h]\|_{L_t^\theta L_x^p} &= \left\| \left\| \int_0^t K_1(t-s)h(s)ds \right\|_p \right\|_{L_t^\theta} \\
&\le \left\| \int_0^t \|K_1(t-s)h(s)\|_p \, ds \right\|_{L_t^\theta} \\
&\le \left\| \int_0^t \left\| \{K_1(t-s) - e^{-\frac{t-s}{2}} W_1(t-s)\}h(s) \right\|_p ds \right\|_{L_t^\theta} \\
&\quad + \left\| \int_0^t \left\| e^{-\frac{t-s}{2}} W_1(t-s)h(s) \right\|_p ds \right\|_{L_t^\theta} \\
&\le C \left\| \int_0^t \frac{\|h(s)\|_{p'}}{|1+t-s|^{\gamma_D}} ds \right\|_{L_t^\theta} + C \left\| \int_0^t e^{-\frac{t-s}{2}} \frac{\|h(s)\|_{\dot{W}^{\mu-1,p'}}}{|t-s|^{\gamma_W}} ds \right\|_{L_t^\theta} \\
&= I_1 + I_2.
\end{aligned}$$

Because of $\mu \ge 1$, $\gamma_D \ge \frac{2}{\theta}$ and $\gamma_W \le \frac{2}{\theta}$, it holds from Hardy-Littlewood-Sobolev's inequality (Proposition 2.6) that

$$I_1 \le C \left\| \int_0^t \frac{\|h(s)\|_{p'}}{|t-s|^{\frac{2}{\theta}}} ds \right\|_{L_t^\theta} \le C\|h\|_{L_t^{\theta'} L_x^{p'}} \le C\|h\|_{L_t^{\theta'} W_x^{\mu-1,p'}}$$

and

$$I_2 \le C \left\| \int_0^t \frac{\|h(s)\|_{\dot{W}^{\mu-1,p'}}}{|t-s|^{\frac{2}{\theta}}} ds \right\|_{L_t^\theta} \le C\|h\|_{L_t^{\theta'} \dot{W}_x^{\mu-1,p'}} \le C\|h\|_{L_t^{\theta'} W_x^{\mu-1,p'}}.$$

Thus we get (3.6).

Next, we prove (3.5). Using Proposition 2.1 and Proposition 2.4, we have

$$
\begin{aligned}
&\|\Phi_0[h]\|_{L^\theta_t L^p_x} = \left\| \left\| \int_0^t K_0(t-s)h(s)ds \right\|_p \right\|_{L^\theta_t} \\
&\le \left\| \int_0^t \|K_0(t-s)h(s)\|_p \, ds \right\|_{L^\theta_t} \\
&\le \left\| \int_0^t \left\| \left\{ K_0(t-s) - e^{-\frac{t-s}{2}} \left(W_0(t-s) + \frac{(t-s)}{8} W_1(t-s) \right) \right\} h(s) \right\|_p ds \right\|_{L^\theta_t} \\
&+ \left\| \int_0^t \left\| e^{-\frac{t-s}{2}} W_0(t-s)h(s) \right\|_p ds \right\|_{L^\theta_t} + \left\| \int_0^t \left\| e^{-\frac{t-s}{2}} \frac{t-s}{8} W_1(t-s)h(s) \right\|_p ds \right\|_{L^\theta_t} \\
&\le C \left\| \int_0^t \frac{\|h(s)\|_{p'}}{|1+t-s|^{\gamma_D+1}} ds \right\|_{L^\theta_t} + C \left\| \int_0^t e^{-\frac{t-s}{2}} \frac{\|h(s)\|_{\dot W^{\mu,p'}}}{|t-s|^{\gamma_W}} ds \right\|_{L^\theta_t} \\
&+ C \left\| \int_0^t e^{-\frac{t-s}{2}} \frac{\|h(s)\|_{\dot W^{\mu-1,p'}}}{|t-s|^{\gamma_W - 1}} ds \right\|_{L^\theta_t} \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

Using the same argument as to estimate I_1 and I_2, it holds that

$$
J_1 \le C\|h\|_{L^{\theta'}_t L^{p'}_x}, \; J_2 \le C\|h\|_{L^{\theta'}_t W^{\mu,p'}_x} \text{ and } J_3 \le C\|h\|_{L^{\theta'}_t W^{\mu-1,p'}_x}.
$$

Thus we get (3.5). This completes the proof of Lemma 3.2. □

Finally, we prove $L^\infty_t L^2_x$-$L^{\theta'}_t L^{p'}_x$ type estimate.

Lemma 3.3. *Suppose* $2 \le p < \infty$, $2 < \theta < \infty$ *and* $\mu \ge 1$ *satisfy* (3.2)-(3.4). *For any* $\nu \ge \frac{\mu}{2}$, *it holds that*

$$
\|\Phi_1[h]\|_E \le C\|h\|_{L^{\theta'}_t W^{\nu,p'}_x}. \tag{3.7}
$$

Proof. Using Proposition 2.3, we have

$$
\|\Phi_1[h]\|_E^2 \le C\{\|\partial_t \Phi_1[h]\|_{L^\theta_t W^{-\nu,p}_x} + \|\Phi_1[h]\|_{L^\theta_t W^{-\nu,p}_x}\}\|h\|_{L^{\theta'}_t W^{\nu,p'}_x}. \tag{3.8}
$$

Because of the definition of Φ_j, it holds that

$$
\partial_t \Phi_1[h](t) = -\frac{1}{2}\Phi_1[h](t) + \Phi_0[h](t).
$$

From Lemma 3.2, it holds that

$$
\begin{aligned}
\|\partial_t \Phi_1[h]\|_{L^\theta_t W^{-\nu,p}_x} &= \|\langle D\rangle^{-\nu}\partial_t \Phi_1[h]\|_{L^\theta_t L^p_x} = \|\partial_t \Phi_1[\langle D\rangle^{-\nu} h]\|_{L^\theta_t L^p_x} \\
&\le C\{\|\Phi_1[\langle D\rangle^{-\nu} h]\|_{L^\theta_t L^p_x} + \|\Phi_0[\langle D\rangle^{-\nu} h]\|_{L^\theta_t L^p_x}\} \\
&\le C\{\|\langle D\rangle^{-\nu} h\|_{L^{\theta'}_t W^{\mu-1,p}_x} + \|\langle D\rangle^{-\nu} h\|_{L^{\theta'}_t W^{\mu,p}_x}\} \le C\|h\|_{L^{\theta'}_t W^{\mu-\nu,p'}_x}
\end{aligned} \tag{3.9}
$$

and

(3.10)
$$\|\Phi_1[h]\|_{L_t^\theta W_x^{-\nu,p}} = \|\Phi_1[\langle D\rangle^{-\nu} h]\|_{L_t^\theta L_x^p} \le C\|\langle D\rangle^{-\nu} h\|_{L_t^{\theta'} W_x^{\mu,p'}} \le C\|h\|_{L_t^{\theta'} W_x^{\mu-\nu,p'}}.$$

Combining (3.8)-(3.10) and $\nu \ge \frac{\mu}{2}$, we get

$$\|\Phi_1[h]\|_E^2 \le C\|h\|_{L_t^{\theta'} W_x^{\mu-\nu,p'}} \|h\|_{L_t^{\theta'} W_x^{\nu,p'}} \le C\|h\|^2_{L_t^{\theta'} W_x^{\nu,p'}}.$$

This means (3.7). This completes the proof of Lemma 3.3. □

Corollary 3.4. *Suppose* $2 \le p < \infty$, $2 < \theta < \infty$ *and* $\mu \ge 1$ *satisfy* (3.2)-(3.4). *Then for any* $2 \le l \le \frac{2d}{d-1}$, *it holds that*

$$\|\Phi_1[h]\|_{L_t^\infty L_x^l} \le C\|h\|_{L_t^{\theta'} W_x^{\mu-1,p'}}.$$

Proof. Using Lemma 3.3 for $\nu = \frac{\mu}{2}$ and Proposition 2.5 for $0 - \frac{d}{l} \le \frac{1}{2} - \frac{d}{2}$, we obtain

$$\begin{aligned}\|\Phi_1[h]\|_{L_t^\infty L_x^l} &\le \|\Phi_1[h]\|_{L_t^\infty W_x^{\frac{1}{2},2}} \le \|\Phi_1[\langle D\rangle^{-\frac{1}{2}} h]\|_{L_t^\infty W_x^{1,2}} \le \|\Phi_1[\langle D\rangle^{-\frac{1}{2}} h]\|_E \\ &\le C\|\langle D\rangle^{-\frac{1}{2}} h\|_{L_t^{\theta'} W_x^{\frac{\mu}{2},p'}} \le C\|h\|_{L_t^{\theta'} W_x^{\frac{\mu}{2}-\frac{1}{2},p'}} \le C\|h\|_{L_t^{\theta'} W_x^{\mu-1,p'}},\end{aligned}$$

where we use $\mu \ge 1$ and $2 \le l \le \frac{2d}{d-1}$. □

Proof of Theorem 1.1

We use the interpolation augments (see for example [3] Chap.2). Let (θ_j, p_j, μ_j) $(j = 1, 2)$ satisfy (1.1)-(1.4). From Lemma 3.1, Lemma 3.2 and Corollary 3.4, we obtain the following three estimates

$$\|\Phi_1[h]\|_{L_t^{\theta_j} L_x^{p_j}} \le C\|h\|_{L_t^1 L_x^2}, \tag{3.11}$$

$$\|\Phi_1[h]\|_{L_t^{\theta_j} L_x^{p_j}} \le C\|h\|_{L_t^{\theta_j'} W_x^{\mu_j-1,p_j'}}, \tag{3.12}$$

$$\|\Phi_1[h]\|_{L_t^\infty L_x^l} \le C\|h\|_{L_t^{\theta_j'} W_x^{\mu_j-1,p_j'}}, \tag{3.13}$$

where $2 \le l \le \frac{2d}{d-1}$ and $\mu_j = \max\left\{1, (d+1)\left(\frac{1}{2} - \frac{1}{p_j}\right)\right\}$. First, we prove (I). Let $2 \le l \le \frac{2d}{d-1}$ and $\sigma \in [0,1]$ satisfy (1.5). Then using Hölder inequality, (3.12) and (3.13), we obtain

$$\|\Phi_1[h]\|_{L_t^{\theta_1} L_x^{p_1}} \le \|\Phi_1[h]\|^\sigma_{L_t^{\theta_2} L_x^{p_2}} \|\Phi_1[h]\|^{1-\sigma}_{L_t^\infty L_x^l} \le C\|h\|_{L_t^{\theta_2'} W_x^{\mu_2-1,p_2'}}.$$

Thus we get (I).

Next, we prove (II). It holds from (3.11) and (3.12) that

$$\|\Phi_1[h]\|_{L_t^{\theta_1}L_x^{p_1}} \leq C\|h\|_{L_t^1 L_x^2}$$

and

$$\|\Phi_1[h]\|_{L_t^{\theta_1}L_x^{p_1}} \leq C\|h\|_{L_t^{\theta_1'}W_x^{\mu_1-1,p_1'}}.$$

From Interpolation theorem (see for example [1] Theorem 6.4.5.), it follows that

$$\|\Phi_1[h]\|_{L_t^{\theta_1}L_x^{p_1}} \leq C\|h\|_{L_t^{\tilde{\theta}'}W_x^{s,\tilde{p}'}}$$

for any $(\tilde{\theta}, \tilde{p})$ which satisfies

$$\frac{1}{\tilde{p}'} = \frac{\tilde{\sigma}}{p_1'} + \frac{1-\tilde{\sigma}}{2}, \quad \frac{1}{\tilde{\theta}'} = \frac{\tilde{\sigma}}{\theta_1'} + \frac{1-\tilde{\sigma}}{1} \quad \text{and} \quad s = \tilde{\sigma}(\mu_1 - 1) + (1-\tilde{\sigma})\cdot 0$$

for some $\tilde{\sigma} \in [0,1]$. Because of the above fact and (1.7), we obtain

$$\|\Phi_1[h]\|_{L_t^{\theta_1}L_x^{p_1}} \leq C\|h\|_{L_t^{\theta_2'}W_x^{\sigma(\mu_1-1),p_2'}}.$$

Thus we get (II). This completes the proof of Theorem 1.1. □

§ 4. Using solvability for the nonlinear problem in the energy class

In this section, we show the existence and uniqueness of solutions for (NDW) in energy class $W^{1,2} \times L^2$. The proof is based on a fixed point argument (see for example [3] §4). In order to make a contraction mapping, we use Theorem 1.1.

For any $T > 0$, the statements up to §3 hold even if we replace $L_t^p([0,\infty))$ with $L_t^p([0,T))$. Indeed we should put $h_T = \chi_T h$ instead of h, where χ_T is indicator function of $[0,T)$. Throughout this section we denote $L_t^p = L_t^p([0,T))$. We define the operator

$$N[u] = K(u_0, u_1) + \Phi_1[F(u)]$$

and the norms

$$\|u\|_X = \|u\|_E + \|u\|_{L_t^8 L_x^8} + \|u\|_{L_t^4 W_x^{\frac{1}{2},4}}$$

and

$$\|u\|_Y = \|u\|_{L_t^8 L_x^8} + \|u\|_{L_t^4 W_x^{\frac{1}{2},4}}.$$

We prepare a linear Strichartz type estimate.

Lemma 4.1. *Let $d = 3$. Then it holds that*

$$\|K(u_0, u_1)\|_X \leq C\|(u_0, u_1)\|_{W^{1,2}\times L^2}, \tag{4.1}$$

where C is an absolute constant.

Proof. From Proposition 2.3 and Lemma 2.8, it holds that

$$\|K(u_0, u_1)\|_E + \|K(u_0, u_1)\|_{L_t^4 L_x^4} \leq C\|(u_0, u_1)\|_{W^{1,2}\times L^2}.$$

Thus it suffices to show

$$\|K(u_0, u_1)\|_{L_t^8 L_x^8} \leq C\|(u_0, u_1)\|_{W^{1,2}\times L^2} \tag{4.2}$$

and

$$\|K(u_0, u_1)\|_{L_t^4 \dot{W}_x^{\frac{1}{2},4}} \leq C\|(u_0, u_1)\|_{W^{1,2}\times L^2}. \tag{4.3}$$

First, we prove (4.2). Using Proposition 2.2 for $(\theta, p, \mu) = (8, 8, 1)$, we obtain

$$\left\| e^{-\frac{t}{2}} \left(W_0(t) + \frac{t}{8} W_1(t) \right) u_0 \right\|_{L_t^8 L_x^8} \leq C\|u_0\|_{W^{1,2}}$$

and

$$\left\| e^{-\frac{t}{2}} W_1(t) \left(\frac{1}{2} u_0 + u_1 \right) \right\|_{L_t^8 L_x^8} \leq C\|(u_0, u_1)\|_{L^2\times L^2}.$$

On the other hand, from the Proposition 2.4 for $p = 8, q' = 2$, it follows that

$$\begin{aligned} &\left\| K_0(t)u_0 - e^{-\frac{t}{2}} \left(W_0(t) + \frac{t}{8} W_1(t) \right) u_0 \right\|_{L_t^8 L_x^8} \\ &\leq C \left\| (1+t)^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{8}\right)-1} \|u_0\|_2 \right\|_{L_t^8} \leq C\|u_0\|_2 \end{aligned}$$

and

$$\begin{aligned} &\left\| K_1(t) \left(\frac{1}{2} u_0 + u_1 \right) - e^{-\frac{t}{2}} W_1(t) \left(\frac{1}{2} u_0 + u_1 \right) \right\|_{L_t^8 L_x^8} \\ &\leq C \left\| (1+t)^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{8}\right)} \|(u_0, u_1)\|_{L^2\times L^2} \right\|_{L_t^8} \leq C\|(u_0, u_1)\|_{L^2\times L^2}. \end{aligned}$$

Hence we calculate

$$
\begin{aligned}
&\|K(u_0,u_1)u_1\|_{L_t^8L_x^8}\\
&\le \left\|K_0(t)u_0 - e^{-\frac{t}{2}}\left(W_0(t)+\frac{t}{8}W_1(t)\right)u_0\right\|_{L_t^8L_x^8}\\
&+\left\|K_1(t)\left(\frac{1}{2}u_0+u_1\right) - e^{-\frac{t}{2}}W_1(t)\left(\frac{1}{2}u_0+u_1\right)\right\|_{L_t^8L_x^8}\\
&+\left\|e^{-\frac{t}{2}}\left(W_0(t)+\frac{t}{8}W_1(t)\right)u_0\right\|_{L_t^8L_x^8}+\left\|e^{-\frac{t}{2}}W_1(t)\left(\frac{1}{2}u_0+u_1\right)\right\|_{L_t^8L_x^8}\\
&\le C\|(u_0.u_1)\|_{W^{1,2}\times L^2},
\end{aligned}
$$

thus we get (4.2).

Next, we prove (4.3). We use a decomposition by Fourier transform (see [5], [11]). Let $\chi \in C^\infty(\mathbb{R}^d_\xi)$ be a cut-off function which satisfies

$$0\le \chi\le 1,\quad \chi(\xi)=1\quad (|\xi|\le 1)\quad \textit{and}\quad \chi(\xi)=0\quad (|\xi|\ge 2).$$

Using Lemma 2.8, we obtain

$$
\begin{aligned}
&\|\chi(D)|D|^{\frac{1}{2}}K(u_0,u_1)\|_{L_t^4L_x^4} = \|K(\chi(D)|D|^{\frac{1}{2}}u_0,\chi(D)|D|^{\frac{1}{2}}u_1)\|_{L_t^4L_x^4}\\
&\le C\|(\chi(D)|D|^{\frac{1}{2}}u_0,\chi(D)|D|^{\frac{1}{2}}u_1)\|_{W^{1,2}\times L^2}\le C\|(u_0,u_1)\|_{L^2\times L^2}.
\end{aligned}
$$

Therefore if the following estimate

$$\|(1-\chi(D))|D|^{\frac{1}{2}}K(u_0,u_1)\|_{L_t^4L_x^4}\le C\|(u_0,u_1)\|_{\dot{W}^{1,2}\times L^2} \tag{4.4}$$

is held, we obtain (4.3). By the fundamental theorem of calculus, it holds that

$$
\begin{aligned}
&(1-\chi(\xi))\cos(t\varphi(\xi)) = (1-\chi(\xi))\left\{\cos(t|\xi|) - t\int_{|\xi|}^{\varphi(\xi)}\sin(t\eta_1)d\eta_1\right\}\\
&=(1-\chi(\xi))\left\{\cos(t|\xi|)+t(|\xi|-\varphi(\xi))\sin(t|\xi|)-\frac{t^2}{2}(|\xi|-\varphi(\xi))^2\cos(t\xi)\right.\\
&\qquad\left.+t^3\int_{|\xi|}^{\varphi(\xi)}\int_{|\xi|}^{\eta_1}\int_{|\xi|}^{\eta_2}\sin(t\eta_3)d\eta_3 d\eta_2 d\eta_1\right\}.
\end{aligned}
$$

By the definition of φ and the Taylor expansion, we have

$$|\xi|-\varphi(\xi) = -\frac{1}{8|\xi|}+O(|\xi|^{-3})\quad (|\xi|\to\infty). \tag{4.5}$$

Hence we obtain

$$(1-\chi(\xi))\left\{\cos(t\varphi(\xi)) - \left(\cos(t|\xi|)+\frac{t\sin(t|\xi|)}{8|\xi|}+\frac{t^2\cos(t|\xi|)}{128|\xi|^2}\right)\right\} = (1-\chi(\xi))R_0, \tag{4.6}$$

where the reminder term $R_0 = R_0(t,\xi)$ satisfies $|R_0(t,\xi)| \le C(1+t)^3/|\xi|^3$ ($t \in [0,\infty)$ and $|\xi| \ge 1$). On the other hand, using (4.6), we also have

$$
\begin{aligned}
&(1-\chi(\xi))\frac{\sin(t\varphi(\xi))}{\varphi(\xi)} = (1-\chi(\xi))\int_0^t \cos(s\varphi(\xi))ds \\
&= (1-\chi(\xi))\left\{\int_0^t \left(\cos(s|\xi|) + \frac{s\sin(s|\xi|)}{8|\xi|} + \frac{s^2\cos(s|\xi|)}{128|\xi|^2}\right) ds + \int_0^t R_0(\xi,s)ds\right\} \\
&= (1-\chi(\xi))\left\{\frac{\sin(t|\xi|)}{|\xi|} - \frac{t\cos(t|\xi|)}{8|\xi|^2}\right\} + (1-\chi(\xi))R_1
\end{aligned}
$$

where $|R_1(t,\xi)| \le C(1+t)^4/|\xi|^3$ ($t \in [0,\infty)$ and $|\xi| \ge 1$). Therefore we get

$$
\begin{aligned}
&(1-\chi(\xi))|\xi|^{\frac{1}{2}}\left\{\cos(t\varphi(\xi)) - \left(\cos(t|\xi|) + \frac{t\sin(t|\xi|)}{8|\xi|} + \frac{t^2\cos(t|\xi|)}{128|\xi|^2}\right)\right\} \\
&= (1-\chi(\xi))\tilde{R}_0(t,\xi)
\end{aligned}
\tag{4.7}
$$

and

$$
(1-\chi(\xi))|\xi|^{\frac{1}{2}}\left\{\frac{\sin(t\varphi(\xi))}{\varphi(\xi)} - \left(\frac{\sin(t|\xi|)}{|\xi|} - \frac{t\cos(t|\xi|)}{8|\xi|^2}\right)\right\} = (1-\chi(\xi))\tilde{R}_1(t,\xi),
\tag{4.8}
$$

where $|\tilde{R}_j(t,\xi)| \le C(1+t)^{3+j}/|\xi|^{\frac{5}{2}}$ ($t \in [0,\infty)$ and $|\xi| \ge 1$) and $(1-\chi)\tilde{R}_j(t,\cdot) \in W^{2,2}(\mathbb{R}^3_\xi)$. By Carleson-Beurling's inequality (see [5] Lemma 3.1), it holds from (4.7) and (4.8) that

$$
\begin{aligned}
&\left\|(1-\chi(D))|D|^{\frac{1}{2}}\left\{K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{tW_1(t)}{8} + \frac{t^2W_0(t)}{128|D|^2}\right)\right\} f\right\|_p \\
&\le Ce^{-\varepsilon t}\|f\|_p
\end{aligned}
\tag{4.9}
$$

and

$$
\left\|(1-\chi(D))|D|^{\frac{1}{2}}\left\{K_1(t) - e^{-\frac{t}{2}}\left(W_1(t) - \frac{tW_0(t)}{8|D|^2}\right)\right\} f\right\|_p \le Ce^{-\varepsilon t}\|f\|_p
\tag{4.10}
$$

for any $1 \le p \le \infty$ and $0 < \varepsilon < \frac{1}{2}$. Furthermore, using Hausdorff inequality (see [1] Theorem 1.2.1), Proposition 2.7, (4.7) and (4.8), we have

$$
\begin{aligned}
&\left\|(1-\chi(D))|D|^{\frac{1}{2}}\left\{K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{tW_1(t)}{8} + \frac{t^2W_0(t)}{128|D|^2}\right)\right\} f\right\|_p \\
&\le C\left\|\mathcal{F}^{-1}\left[(1-\chi(\xi))e^{-\frac{t}{2}}\tilde{R}_0(t,\xi)\right]\right\|_p \|f\|_1 \\
&\le C\|(1-\chi(\cdot))e^{-\frac{t}{2}}\tilde{R}_0(t,\cdot)\|_{p'}\|f\|_1 \le Ce^{-\varepsilon t}\|f\|_1
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\left\| (1-\chi(D))|D|^{\frac{1}{2}} \left\{ K_1(t) - e^{-\frac{t}{2}} \left(W_1(t) - \frac{tW_0(t)}{8|D|^2} \right) \right\} f \right\|_p \\
&\le C \left\| \mathcal{F}^{-1} \left[(1-\chi(\xi)) e^{-\frac{t}{2}} \tilde{R}_1(t,\xi) \right] \right\|_p \|f\|_1 \\
&\le C \|(1-\chi(\cdot)) e^{-\frac{t}{2}} \tilde{R}_1(t,\cdot)\|_{p'} \|f\|_1 \le C e^{-\varepsilon t} \|f\|_1 \qquad (4.12)
\end{aligned}
$$

for any $2 \le p < 6$ and $0 < \varepsilon < \frac{1}{2}$. By the interpolation theorem (see [1] Chap.5) with (4.9)-(4.12), we obtain

$$
\begin{aligned}
&\left\| (1-\chi(D))|D|^{\frac{1}{2}} \left\{ K_0(t) - e^{-\frac{t}{2}} \left(W_0(t) + \frac{tW_1(t)}{8} + \frac{t^2 W_0(t)}{128|D|^2} \right) \right\} u_0 \right\|_4 \\
&\le C e^{-\varepsilon t} \|u_0\|_2 \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| (1-\chi(D))|D|^{\frac{1}{2}} \left\{ K_1(t) - e^{-\frac{t}{2}} \left(W_1(t) - \frac{tW_0(t)}{8|D|^2} \right) \right\} \left(\frac{1}{2} u_0 + u_1 \right) \right\|_4 \\
&\le C e^{-\varepsilon t} \|(u_0, u_1)\|_{L^2 \times L^2} \qquad (4.14)
\end{aligned}
$$

for any $0 < \varepsilon < \frac{1}{2}$. On the other hand, using Proposition 2.2, we obtain

$$
\begin{aligned}
&\left\| (1-\chi(D))|D|^{\frac{1}{2}} \left\{ e^{-\frac{t}{2}} \left(W_0(t) + \frac{tW_1(t)}{8} + \frac{t^2 W_0(t)}{128|D|^2} \right) \right\} u_0 \right\|_4 \qquad (4.15) \\
&\le C e^{-\varepsilon t} \left\{ \|(1-\chi(D))u_0\|_{\dot{W}_x^{1,2}} + \|(1-\chi(D))u_0\|_{L_x^2} + \|(1-\chi(D))u_0\|_{\dot{W}_x^{-1,2}} \right\} \\
&\le C e^{-\varepsilon t} \|u_0\|_{\dot{W}^{1,2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| (1-\chi(D))|D|^{\frac{1}{2}} \left\{ e^{-\frac{t}{2}} \left(W_1(t) - \frac{tW_0(t)}{8|D|^2} \right) \right\} \left(\frac{1}{2} u_0 + u_1 \right) \right\|_4 \qquad (4.16) \\
&\le C e^{-\varepsilon t} \{ \|((1-\chi(D))u_0, (1-\chi(D))u_1)\|_{L^2 \times L^2} \\
&\qquad + \|((1-\chi(D))u_0, (1-\chi(D))u_1)\|_{\dot{W}^{-1,2} \times \dot{W}^{-1,2}} \} \\
&\le C e^{-\varepsilon t} \|(u_0, u_1)\|_{L^2 \times L^2}
\end{aligned}
$$

for any $0 < \varepsilon < \frac{1}{2}$. Combining (4.13)-(4.16), we obtain (4.4). This completes the proof of Lemma 4.1.

□

Proof of Theorem 1.2

Let $d = 3$ and $\|(u_0, u_1)\|_{W^{1,2} \times L^2} \le A$. We define the complete metric space

$$
X_1 = \left\{ u \in C^1([0,T); L_x^2) \cap C([0,T); W_x^{1,2}) \cap L_t^8 L_x^8 \cap L_t^4 W_x^{\frac{1}{2},4}; \|u\|_X \le M \right\}
$$

with the metric $d(v,w) = \|v - w\|_Y$, where $M = CA + 1$ and C is given in Lemma 4.1.

First, we prove N is a map from X_1 to itself. Let $u \in X_1$. From Proposition 2.5, it holds

$$\|\Phi_1[F]\|_{L_t^8 L_x^8} \le C\|\Phi_1[F]\|_{L_t^8 W_x^{\frac{1}{2},\frac{24}{7}}}.$$

Then using Theorem 1.1 (I) for $l = 3$, $(\theta_1, p_1) = (4,4), (8, \frac{24}{7})$ and $(\theta_2, p_2) = (4,4)$, we obtain

$$\|\Phi_1[F]\|_{L_t^8 L_x^8} + \|\Phi_1[F]\|_{L_t^4 W_x^{\frac{1}{2},4}} \le C\|F\|_{L_t^{\frac{4}{3}} W_x^{\frac{1}{2},\frac{4}{3}}}.$$

Moreover using Lemma 3.3, we have

$$\|\Phi_1[F]\|_E \le C\|F\|_{L_t^{\frac{4}{3}} W_x^{\frac{1}{2},\frac{4}{3}}}.$$

Using the above estimates and Lemma 4.1, we have

$$\|N[u]\|_X \le C\|(u_0, u_1)\|_{W^{1,2}\times L^2} + C\||u|^\alpha\|_{L_t^{\frac{4}{3}} W_x^{\frac{1}{2},\frac{4}{3}}}. \tag{4.17}$$

From the chain rule for fractional derivative (see §5 Lemma 5.1), it holds from $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ that

$$\||u|^\alpha\|_{W^{\frac{1}{2},\frac{4}{3}}} \le C\||u|^{\alpha-1}\|_2 \|u\|_{W^{\frac{1}{2},4}}.$$

Because of $3 < \alpha < 5$, we can put a $\lambda_1 \in (0,1)$ such that it satisfies

$$\frac{1}{2(\alpha-1)} = \frac{\lambda_1}{2} + \frac{1-\lambda_1}{8}, \tag{4.18}$$

then we have

$$\|u\|^{\alpha-1}_{L_t^{2(\alpha-1)} L_x^{2(\alpha-1)}} \le \|u\|^{\lambda_1(\alpha-1)}_{L_t^2 L_x^2} \|u\|^{(1-\lambda_1)(\alpha-1)}_{L_t^8 L_x^8}.$$

Combining the above estimates and Hölder inequality, we obtain

$$\begin{aligned}
\||u|^\alpha\|_{L_t^{\frac{4}{3}} W_x^{\frac{1}{2},\frac{4}{3}}} &\le \||u|^{\alpha-1}\|_{L_t^2 L_x^2} \|u\|_{L_t^4 W_x^{\frac{1}{2},4}} = \|u\|^{\alpha-1}_{L_t^{2(\alpha-1)} L_x^{2(\alpha-1)}} \|u\|_{L_t^4 W_x^{\frac{1}{2},4}} \\
&\le C\|u\|^{\lambda_1(\alpha-1)}_{L_t^2 L_x^2} \|u\|^{(1-\lambda_1)(\alpha-1)}_{L_t^8 L_x^8} \|u\|_{L_t^4 W_x^{\frac{1}{2},4}} \\
&\le CT^{\frac{\lambda_1}{2}(\alpha-1)} \|u\|^{\lambda_1(\alpha-1)}_{L_t^\infty L_x^2} \|u\|^{(1-\lambda_1)(\alpha-1)}_{L_t^8 L_x^8} \|u\|_{L_t^4 W_x^{\frac{1}{2},4}} \\
&\le CT^{\frac{\lambda_1}{2}(\alpha-1)} \|u\|_X^\alpha \le CT^{\frac{\lambda_1}{2}(\alpha-1)} M^\alpha.
\end{aligned} \tag{4.19}$$

Thus it holds from (4.17) that

$$\|N[u]\|_X \le CA + CT^{\frac{\lambda_1}{2}(\alpha-1)} M^\alpha. \tag{4.20}$$

We choose $0 < T \le 1$ such that $CT^{\frac{\lambda_1}{2}(\alpha-1)} M^\alpha \le 1$, then $N[u] \in X_1$.

Next, we prove that N has the contraction property. Let $v, w \in X_1$. Using Theorem 1.1 and Proposition 2.5, we have

$$\|N[v]-N[w]\|_{L_t^8L_x^8}+\|N[v]-N[w]\|_{L_t^4W_x^{\frac{1}{2},4}}\le C\|F(v)-F(w)\|_{L_t^{\frac{4}{3}}W_x^{\frac{1}{2},\frac{4}{3}}}.$$

Lemma 5.1 with $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}=\frac{3}{8}+\frac{1}{8}+\frac{1}{4}$ implies that

$$\begin{aligned}
&d(N[v],N[w])\le C\|F(v)-F(w)\|_{L_t^{\frac{4}{3}}W_x^{\frac{1}{2},\frac{4}{3}}}\\
&\le C\left\|(\||v|^{\alpha-1}\|_2+\||w|^{\alpha-1}\|_2)\|v-w\|_{W^{\frac{1}{2},4}}\right\|_{L_t^{\frac{4}{3}}}\\
&+C\left\|(\||v|^{\alpha-2}\|_{\frac{8}{3}}+\||w|^{\alpha-2}\|_{\frac{8}{3}})(\|v\|_{W^{\frac{1}{2},4}}+\|w\|_{W^{\frac{1}{2},4}})\|v-w\|_8\right\|_{L_t^{\frac{4}{3}}}\\
&\le C\left(\|v\|_{L_t^2L_x^2}^{\lambda_1(\alpha-1)}\|v\|_{L_t^8L_x^8}^{(1-\lambda_1)(\alpha-1)}+\|w\|_{L_t^2L_x^2}^{\lambda_1(\alpha-1)}\|w\|_{L_t^8L_x^8}^{(1-\lambda_1)(\alpha-1)}\right)\|v-w\|_{L_t^4W_x^{\frac{1}{2},4}}\\
&\quad+C\left(\|v\|_{L_t^{\frac{8}{3}(\alpha-2)}L_x^{\frac{8}{3}(\alpha-2)}}^{\alpha-2}+\|w\|_{L_t^{\frac{8}{3}(\alpha-2)}L_x^{\frac{8}{3}(\alpha-2)}}^{\alpha-2}\right)\\
&\qquad\times\left(\|v\|_{L_t^4W_x^{\frac{1}{2},4}}+\|w\|_{L_t^4W_x^{\frac{1}{2},4}}\right)\|v-w\|_{L_t^8L_x^8},
\end{aligned}$$

where $\lambda_1\in(0,1)$ is defined by (4.18). Because of $3<\alpha<5$, there exists $\lambda_2\in(0,1)$ such that

$$\frac{3}{8(\alpha-2)}=\frac{\lambda_2}{2}+\frac{1-\lambda_2}{8} \tag{4.21}$$

which satisfies

$$\begin{aligned}
\|v\|_{L_t^{\frac{8}{3}(\alpha-2)}L_x^{\frac{8}{3}(\alpha-2)}}^{\alpha-2}&\le\|v\|_{L_t^2L_x^2}^{\lambda_2(\alpha-2)}\|v\|_{L_t^8L_x^8}^{(1-\lambda_2)(\alpha-2)}\\
&\le T^{\frac{\lambda_2}{2}(\alpha-2)}\|v\|_{L_t^\infty L_x^2}^{\lambda_2(\alpha-2)}\|v\|_{L_t^8L_x^8}^{(1-\lambda_2)(\alpha-2)}.
\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}
&d(N[v],N[w])\\
&\le CT^{\frac{\lambda_1}{2}(\alpha-1)}\left(\|v\|_{L_t^\infty L_x^2}^{\lambda_1(\alpha-1)}\|v\|_{L_t^8L_x^8}^{(1-\lambda_1)(\alpha-1)}+\|w\|_{L_t^\infty L_x^2}^{\lambda_1(\alpha-1)}\|w\|_{L_t^8L_x^8}^{(1-\lambda_1)(\alpha-1)}\right)\\
&\qquad\times\|v-w\|_{L_t^4W_x^{\frac{1}{2},4}}\\
&+CT^{\frac{\lambda_2}{2}(\alpha-2)}\left(\|v\|_{L_t^\infty L_x^2}^{\lambda_2(\alpha-2)}\|v\|_{L_x^8L_x^8}^{(1-\lambda_2)(\alpha-2)}+\|w\|_{L_t^\infty L_x^2}^{\lambda_2(\alpha-2)}\|w\|_{L_x^8L_x^8}^{(1-\lambda_2)(\alpha-2)}\right)\\
&\qquad\times\left(\|v\|_{L_t^4W_x^{\frac{1}{2},4}}+\|w\|_{L_t^4W_x^{\frac{1}{2},4}}\right)\|v-w\|_{L_t^8L_x^8}\\
&\le CM^{\alpha-1}\left(T^{\frac{\lambda_1}{2}(\alpha-1)}+T^{\frac{\lambda_2}{2}(\alpha-2)}\right)d(v,w).
\end{aligned} \tag{4.22}$$

Then we choose small $T > 0$ such that $CM^{\alpha-1}(T^{\frac{\lambda_2}{2}(\alpha-1)} + T^{\frac{\lambda_2}{2}(\alpha-2)}) \le \frac{1}{2}$ again, we get $d(N[v], N[w]) \le \frac{1}{2} d(v, w)$.

Therefore N becomes a contraction mapping on X_1. By Banach's fixed point theorem, N has a unique fixed point $u \in X_1$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3

Let $d = 3$, $\alpha = 5$, $0 < T \le \infty$ and $\|K(u_0, u_1)\|_{L^8_t L^8_x} \le \delta$. We put $A > 0$ satisfies $\|(u_0, u_1)\|_{W^{1,2} \times L^2} \le A$ and define the complete metric space

$$X_2 = \left\{ u \in L^8_t L^8_x \cap L^4_t W^{\frac{1}{2},4}_x;\ \|u\|_{L^8_t L^8_t} \le a \text{ and } \|u\|_{L^4_t W^{1,4}_x} \le b \right\}$$

with the metric $d(v, w) = \|v - w\|_Y$, where $a, b > 0$ are determined later.

For $u \in X_2$, we use the same argument as in the proof of (4.19). Noting $\lambda_1 = 0$ in (4.18), we have

$$\begin{aligned} \|N[u]\|_{L^8_t L^8_x} &\le \|K(u_0, u_1)\|_{L^8_t L^8_x} + C \| |u|^5 \|_{L^{\frac{4}{3}}_t W^{\frac{1}{2},\frac{4}{3}}_x} \\ &\le \delta + \tilde{C}_0 \|u\|^4_{L^8_t L^8_x} \|u\|_{L^4_t W^{\frac{1}{2},4}_x} \le \delta + \tilde{C}_0 a^4 b \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \|N[u]\|_{L^4_t W^{\frac{1}{2},4}_x} &\le \|K(u_0, u_1)\|_{L^4_t W^{\frac{1}{2},4}_x} + C \| |u|^5 \|_{L^{\frac{4}{3}}_t W^{\frac{1}{2},\frac{4}{3}}_x} \\ &\le \tilde{C}_1 A + \tilde{C}_2 \|u\|^4_{L^8_t L^8_x} \|u\|_{L^4_t W^{1,4}_x} \le \tilde{C}_1 A + \tilde{C}_2 a^4 b, \end{aligned} \tag{4.24}$$

where the constants $\tilde{C}_j$ $(j = 0, 1, 2)$ are independent of δ, A, a and b. We take $b = 2\tilde{C}_1 A$, $a = 2\delta$ and choose δ which satisfies

$$\tilde{C}_0 a^3 b \le \frac{1}{2} \quad \text{and} \quad \tilde{C}_2 a^4 \le \frac{1}{2},$$

then it holds from (4.23) and (4.24) that $N[u] \in X_2$.

Next, we prove that N has the contraction property. Let $v, w \in X_2$. By the same argument as for the proof of (4.22), we have

$$\begin{aligned} & d(N[v], N[w]) \\ & \le C \left(\|v\|^4_{L^8_t L^8_x} + \|w\|^4_{L^8_t L^8_x} \right) \|v - w\|_{L^4_t W^{\frac{1}{2},4}_x} \\ & + C \left(\|v\|^3_{L^8_t L^8_x} + \|w\|^3_{L^8_t L^8_x} \right) \left(\|v\|_{L^4_t W^{\frac{1}{2},4}_x} + \|w\|_{L^4_t W^{\frac{1}{2},4}_x} \right) \|v - w\|_{L^8_t L^8_x} \\ & \le \left(\tilde{C}_3 a^4 + \tilde{C}_4 a^3 b \right) d(v, w), \end{aligned}$$

where we remark that $\lambda_2 = 0$ in (4.21) and $\tilde{C}_j$ $(j = 3, 4)$ are independent of δ, A, a and b. Remember $\delta = a/2$, we choose δ again such that $\tilde{C}_3 a^4 + \tilde{C}_4 a^3 b \le \frac{1}{2}$, then we get $d(N[v], N[w]) \le \frac{1}{2} d(v, w)$.

Therefore N becomes a contraction mapping on X_2. By Banach's fixed point theorem, N has a unique fixed point $u \in X_2$. Using Proposition 2.3 and Lemma 3.3, we have

$$\|N[u]\|_E \le C\|(u_0, u_1)\|_{W^{1,2}\times L^2} + \||u|^5\|_{L_t^{\frac{4}{3}} W_x^{\frac{1}{2},\frac{4}{3}}} \le CA + Ca^4 b.$$

for any $u \in X_2$. This means that the solution $u = N[u]$ belongs $C^1([0,T); L^2) \cap C^0([0,T); W^{1,2})$. This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4

First, we prove (i). By Lemma 4.1, it holds that

$$\|K(u_0, u_1)\|_{L_t^8 L_x^8} \le C\|(u_0, u_1)\|_{W^{1,2}\times L^2},$$

thus $\|K(u_0, u_1)\|_{L_t^8 L_x^8} \to 0$ as $T \to 0$. We choose small $T > 0$ which satisfies $\|K(u_0, u_1)\|_{L_t^8 L_x^8} \le \delta$, then we obtain a unique local solution to (DW) in $C^1([0,T); L^2) \cap C^0([0,T); W^{1,2})$ from Theorem 1.3.

Next we prove (ii). Let be $L_t^8 = L_t^8([0,\infty))$. Using Lemma 4.1 and the assumption of (ii), it holds that

$$\|K(u_0, u_1)\|_{L_t^8 L_x^8} \le C\|(u_0, u_1)\|_{W^{1,2}\times L^2} \le C\tilde{\delta}.$$

Therefore we choose $\tilde{\delta} > 0$ satisfies $C\tilde{\delta} \le \delta$, then from Theorem 1.3 we obtain a unique global solution to (NDW) in $C^1([0,\infty); L^2) \cap C^0([0,\infty); W^{1,2})$. This completes the proof of Theorem 1.4.

$\square$

§ 5. Appendix : The chain rule for fractional derivative

We show the chain rule for fractional derivative in Sobolev space $W^{s,p}$, where $s \in (0,1)$ and $p \in (1,\infty)$. The following lemma is introduced in [8], however not proved. Thus we give a proof in this section.

Lemma 5.1. *Assume that F is a C^2 map. Furthermore F satisfies $F(0) = F'(0) = 0$ and that for any $\tau \in [0,1]$ and $a, b \in \mathbb{R}$*

$$|F'(\tau a + (1-\tau)b)| \le \mu_1(\tau)(|F'(a)| + |F'(b)|)$$

and

$$|F''(\tau a + (1-\tau)b)| \le \mu_2(\tau)(|F''(a)| + |F''(b)|),$$

where $\mu_j \in L^1_\tau([0,1])$ $(j = 1,2)$. *Then for* $s \in (0,1)$ *and* $p \in (1,\infty)$*, it holds that*

$$\|F(u)\|_{W^{s,p}} \le C\|F'(u)\|_{p_1}\|u\|_{W^{s,p_2}}, \tag{5.1}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_j \in (1,\infty)$ $(j = 1,2)$, *and*

$$\begin{aligned} &\|F(v) - F(w)\|_{W^{s,p}} \\ &\le C(\|F'(v)\|_{p_1} + \|F'(w)\|_{p_1})\|v - w\|_{W^{s,p_2}} \\ &+ C(\|F''(v)\|_{r_1} + \|F''(w)\|_{r_1})(\|v\|_{W^{s,r_2}} + \|w\|_{W^{s,r_2}})\|v - w\|_{r_3}, \end{aligned} \tag{5.2}$$

where $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $r_j \in (1,\infty)$ $(j = 1,2,3)$.

Note that $F(u) = |u|^{\alpha-1}u$ $(\alpha \ge 2)$ satisfies the assumptions. It is clear that even if we replace F which satisfies the assumptions, Theorem 1.2-1.4 are verified.

We introduce the product lemma and the composition lemma, for the proof, see [17] Chap.2.

Lemma 5.2. *For any* $s > 0$ *and* $p \in (1,\infty)$*, we have*

$$\|fg\|_{W^{s,p}} \le C\|f\|_{q_1}\|g\|_{W^{s,q_2}} + C\|f\|_{W^{s,r_1}}\|g\|_{r_2}, \tag{5.3}$$

where

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}, \quad q_2, r_2 \in (1,\infty), \quad q_1, r_1 \in (1,\infty].$$

Lemma 5.3. *Assume* F *is a* C^1 *map, satisfying* $F(0) = 0$ *and*

$$|F'(\tau v + (1-\tau)w)| \le \mu(\tau)(|F'(v)| + |F'(w)|),$$

where $\mu \in L^1_\tau([0,1])$. *Then for any* $s \in (0,1)$ *and* $p \in (1,\infty)$*, we have*

$$\|F(u)\|_{W^{s,p}} \le C\|F'(u)\|_{p_1}\|u\|_{W^{s,p_2}},$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 \in (1,\infty], \quad p_2 \in (1,\infty).$$

Proof of Lemma 5.1

The inequality (5.1) holds from Lemma 5.3. Thus we should prove (5.2). Let put $G(v,w) = \int_0^1 F'(\tau v + (1-\tau)w)d\tau$. Then we have

$$F(v) - F(w) = G(v,w)(v - w).$$

Thus using Lemma 5.2, we obtain

$$\|F(v)-F(w)\|_{W^{s,p}} \le C\|G(v,w)\|_{p_1}\|v-w\|_{W^{s,p_2}} + C\|G(v,w)\|_{W^{s,r}}\|v-w\|_{r_3},$$

where $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ and $\frac{1}{p}=\frac{1}{r}+\frac{1}{r_3}$. From the assumptions and Lemma 5.3, it holds that

$$\|G(v,w)\|_{p_1} \le \int_0^1 \|F'(\tau v+(1-\tau)w)\|_{p_1}d\tau \le C(\|F'(v)\|_{p_1}+\|F'(w)\|_{p_1})$$

and

$$\begin{aligned}\|G(v,w)\|_{W^{s,r}} &\le \int_0^1 \|F'(\tau v+(1-\tau)w)\|_{W^{s,r}}d\tau \\ &\le \int_0^1 \|F''(\tau v+(1-\tau)w)\|_{r_1}\|\tau v+(1-\tau)w\|_{W^{s,r_2}}d\tau \\ &\le C(\|F''(v)\|_{r_1}+\|F''(w)\|_{r_1})(\|v\|_{W^{s,r_2}}+\|w\|_{W^{s,r_2}}).\end{aligned}$$

Combining the above estimates, we obtain (5.1). This completes the proof of Lemma 5.1. □

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